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Phase Transitions in Ising Model Defined on Complex Networks

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Abstract

In this work, we consider an Ising model which allows spin-spin interaction in the systems. We assume that two-level quantum systems are randomly located in *N* nodes of a complex annealed scale-free network described by the Barabasi-Albert model. It is defined by the power-law degree distribution of nodes. We consider the mean-field approach to the system described by the Ising Hamiltonian. At a certain level, the system is totally characterized by the order parameter S_z. It contains a critical inverse temperature β, which depends on parameter ζ_2 as the ratio of the second to the first moment of the degree distribution. We have found that for ζ_2, that exceeds its critical value $\zeta_{2,c}$, high temperature phase transition occurs that can be explained by the hubs and clusters which appear in scale-free networks.

Keywords: Ising model; Phase transitions; Complex networks; Microcavities; Nanomaterials

1. INTRODUCTION

During recent years, due to developing quantum technologies, there has been a growing interest in the study of phase transitions (PTs), as well as the impact of various nonlinear effects on it [1,2].

The nonlinearity, as usual, is really small even in an interacting gas of particles. However, as will be demonstrated in this work, if the particles are "packed" in a special way, such nonlinearity can play a very important role.

The Ising model (IM) presented in this work possess nonlinear features due to spin-spin interaction. Spins are localized in each node of the graph that leads to inhomogeneous spin-spin interaction across the network. Thus, during the propagation of an external magnetic field, the features of the network — such as the degree distribution function of nodes, which is determined by the type of structure — are still taken into account.

The IM is intensively used in many areas of physics [3] and beyond [4–6]. Typically, the collective spin component $S_z = \frac{1}{N}\sum_i \sigma_i^z$ represents order parameter for various Ising-like models [3]; σ_i^z is *z* component of *i*-th particle spin, *N* is a number of particles. The S_z demonstrates second order PT from paramagnetic (non-ordered state with $S_z = 0$) to ferromagnetic (fully ordered state with $|S_z| = 1$).

In this work we consider a general approach to the problem of PTs arising in the network structure and caused by the network finite size effect and nonlinear effects. We investigated the influence of the node degree distribution moments on the PTs in a network structure with a power distribution of degrees of nodes using the adapted Barabasi-Albert algorithm.

We used the mean field approximation for the IM defined on graph structures and obtained an equation for one of the order parameters of the system. This expression allows us to establish the condition of the PT from the paramagnetic state to the ferromagnetic one. We obtained critical values of PT temperatures with an external magnetic field and without one. As a result, we are going to establish a clear relationship between the temperature of the system and the statistical properties of the network.

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2. METHODS. THE ISING MODEL

We started with the IM defined on annealed networks. It was assumed that two-level quantum systems are randomly located in N nodes of a complex network (see Fig. 1). The Hamiltonian of the system is given as [7]:

$$H = -\sum_{ij} J_{ij} \sigma_i^z \sigma_j^z - \frac{1}{2}\sum_i h_i \sigma_i^z, \quad i, j = 1, ..., N, \tag{1}$$

where h_i is a parameter characterizing the interaction of i-th spin with an external magnetic field. The first term in Eq. (1) characterizes interaction between spins and is described by J_{ij} parameter. In fact, J_{ij} determines the topology of the interaction of spins. For the sake of simplicity, we set $h_i = h$ for arbitrary i (hereafter we put the Planck and Boltzmann constants: $\hbar = 1$, $k_B = 1$, for brevity).

In quantum optics Hamiltonian (1) may be realized by means of set of coupled microcavities (micropillars) which contain quantum dots as a two-level (spin-like) systems (TLS) [8]. Since microcavities and quantum dots pose various decoherence mechanisms we assume that strong coupling between TLS and quantized field is fulfilled [9]. Such a condition allows to consider whole system as thermodynamically equilibrium, as it is assumed with exciton-polariton Bose-Einstein condensates; see, e.g., Ref. [10].

3. COMPLEX NETWORK STRUCTURES

Networks are widely distributed in nature and used in various technical applications and society, thus attracting attention of researchers from various fields of science [11–14]. A system with a network architecture is displayed as a graph, which is a convenient way to determine the relationship between a group of elements. It consists of a set of objects called nodes connected by links (edges). Graphs are quite useful because they represent a mathematical model of network structures. The structure (configuration) of the connections between them defines the topology (architecture) of networks. In real networks, the topology of links is usually irregular, but at the same time it is not random. Networks with such features are usually called complex networks.

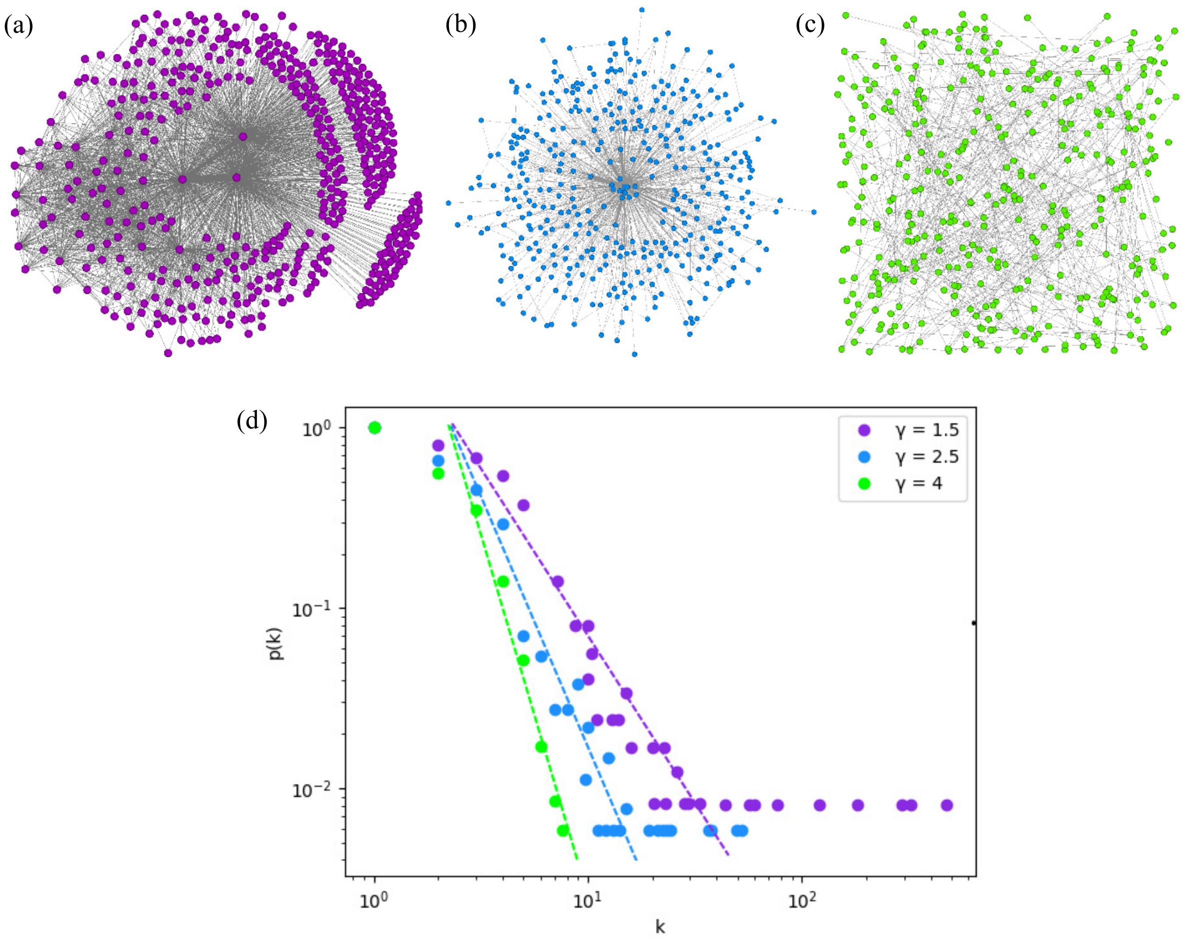

Fig. 1. Power-law degree distribution networks for (a) γ = 1.5, (b) γ = 2.5, (c) γ = 4, which correspond to anomalous, scale-free and random regimes, respectively, (d) power-law degree distributions in a logarithmic scale for the networks given in (a–c). The number of nodes is $N = 500$.

Typically, a network is defined as an ensemble consisting of a finite number of N spins 1/2 (such as electrons) placed in nodes [15], where each particle occupies only one node. The topology is defined by J_{ij} parameter, which stores information about the graph structure. In this work we use the annealed network approach, which assumes a weighted, fully connected graph model. We recast parameter J_{ij} that indicates the coupling between the nodes in Eq. (1) through probability p_{ij} as $J_{ij} = Jp_{ij}$, where J is a constant, and p_{ij} is the probability for two nodes i and j to be connected:

$$p_{ij} = \frac{k_i k_j}{N\langle k\rangle}, \tag{2}$$

where k_i is i-th node degree, which indicates an expected number of the node neighbors and is taken from distribution $p(k)$; $\langle k\rangle = \frac{1}{N}\Sigma k_{ij}$ is an average degree. Noteworthy, the annealed network approach is valid for $p_{ij} << 1$ and large enough N [16]. Thus, the strength of two spins interaction J_{ij} is a variable parameter and depends on particular network characteristics; it is greater for two pairs of nodes with the highest k coefficient.

In practice, different approaches provide an explanation for a real-world network topology [17]. Such networks may exhibit the power-law degree distribution. Since the number of nodes is large enough ($N >> 1$) we are interested in network structures, which admit continuous degree distribution of nodes $p(k)$ determining the probability that a randomly selected node has a certain number of connections k.

This work considers a scale-free network architecture described by the Barabasi-Albert model [17]. It is defined by the power distribution of the degree of nodes as:

$$p(k) = \frac{(\gamma - 1)k_{min}^{\gamma-1}}{k^{\gamma}}, \tag{3}$$

where γ is the exponent, and k_{min} is a minimal connectivity found in the network with given distribution.

An important feature of a scale-free network is the presence of hubs. The largest node is described by the degree k_{max}, and obeys a condition called the natural cutoff:

$$\int_{k_{max}}^{+\infty} p(k)\,dk = \frac{1}{N}. \tag{4}$$

It can be used if the network with N nodes possess more than one node with $k > k_{max}$. From Eq. (4) we obtain

$$k_{max} = k_{min} N^{\frac{1}{\gamma-1}}. \tag{5}$$

Fig. 1d shows probability distribution function (3) plotted in a logarithm scale for the networks which are shown in Figs. 1a–c. The node degree fluctuations grow at $\gamma \le 3$. Hubs in Fig. 1d appear as colored dots in the upper right corner of the plot.

The number of hubs and their size dramatically increase with vanishing γ in the anomalous regime where $k_{max} / k_{min} > N$ (see Fig. 1a and the magenta line in Fig. 1d).

The main characteristics of the network architecture may be characterized by means of the first ($\langle k\rangle$) and the n-th ($\langle k^n\rangle$) moments. The n-th moment is calculated as:

$$K^{(n)} = \langle k^n\rangle = \int_{k_{min}}^{k_{max}} k^n p(k)\,dk, \tag{6}$$

where n is a positive integer. In this work, we are interested in the first and normalized n-th order (n = 2,3,4) degree correlation functions which are defined as

$$\zeta_n = \frac{K^{(n)}}{K^{(1)}}, \quad n = 2,3,4. \tag{7}$$

Eq. (7) defines the main statistical values for the selected network.

Remarkably, the n-th order correlation functions defined in (6) diverge at $\gamma = 1$. On the contrary, their combination $\tau \equiv [K^{(1)}]^2 K^{(4)} / [K^{(2)}]^3$ remains finite. According to the network theory [13], the power distribution has the following regimes. In anomalous ($1 < \gamma < 2$) and scale-free ($2 < \gamma < 3$) regimes scale-free network possesses a set of topological features (for example, clusters and hubs) that support ordered state for IM; $\gamma > 3$ corresponds to random regime. It is known, that for random regime the difference in k and ζ_2 disappears [12,13].

4. PHASE TRANSITION

In this work the mean-field approach to the system described by the Ising Hamiltonian (1) was considered. Such an approach presumes calculation of partition function $Z(T) = \mathrm{Tr}\left[e^{-H/T}\right]$ with Eq. (1), neglecting quantum correlations which occur between the spins; T is a temperature parameter that is relevant to whole ensemble of spins; see Refs. [3,7]. At certain level, the system is totally characterized by the order parameter S_z defined as

$$S_z = \frac{1}{N\langle k\rangle}\sum_i k_i \langle\sigma_i^z\rangle, \tag{8}$$

that represents a weighted average spin component [7]. Collective variable S_z obeys self-consistent equation:

$$S_z = \frac{1}{\langle k\rangle}\int_{k_{min}}^{k_{max}} k\,p(k)\tanh\left[\frac{\beta}{2}\left(4JS_z k + h\right)\right]dk, \tag{9}$$

where $\beta \equiv 1/T$ is inverse temperature. Eq. (9) uses integral form instead of summation.

At high temperatures and non-zero external field, Eq. (9) admits non-zero solution for collective spin component S_z. This solution corresponds to some ferromagnetic phase with $S_z \neq 0$. Critical inverse temperature β_c, which provides this solution, is determined from Eq. (9) and obtained as

$$\beta_c = \frac{1}{2J\zeta_2}, \qquad (10)$$

where parameter ζ_2 is defined in Eq. (7).

Simultaneously, large number of hubs can manifest the activity of spins. In this limit ζ_2 is also large enough and corresponds to high temperature of PT T_c. Critical temperature T_c becomes too large ($\beta_c \simeq 0$) in the vicinity of $\gamma = 1$ where ζ_2 rapidly increases; see Fig. 2a. In the opposite case, for large degree exponent γ parameter ζ_2 approaches k_{min} and critical temperature becomes $T_c \propto k_{min}$.

Remarkably, similar arguments are still true in the mean-field approximation if we neglect degree correlations in the network and put in Eq. (10) $\zeta_2 \simeq \langle k \rangle$, which leads to

$$T_c \simeq 2J\langle k \rangle. \qquad (11)$$

Eqs. (10), (11) play an important role in PT occurring in the finite-size IM. They establish a strong connection between the system temperature and network statistical features. For a given temperature T the critical value $\zeta_{2,c}$ is determined from (10) as

$$\zeta_{2,c} = \frac{T}{2J}. \qquad (12)$$

Then, we approximate $\tanh(x)$ function in Eq. (9) as $\tanh(x) \approx x - \frac{1}{3}x^3$, where $x = \frac{\beta}{2}(4JS_z k + h)$. In this limit Eq. (9) can be represented in the form

$$\alpha_1 S_z - \alpha_2 S_z^3 + \frac{h}{2T} = 0, \qquad (13)$$

where coefficients α_1 and α_2 are defined as:

$$\alpha_1 = \frac{\zeta_2}{\zeta_{2,c}} - 1, \quad \alpha_2 = \frac{\tau}{3}\frac{\zeta_2^3}{\zeta_{2,c}^3}. \qquad (14)$$

From Eq. (13) in the vicinity of critical point $\zeta_2 = \zeta_{2,c}$ and for $h = 0$ we get:

$$S_z = \left[\frac{3}{\tau}\left(\frac{\zeta_2}{\zeta_{2,c}} - 1\right)\right]^{1/2}, \qquad (15)$$

where τ is defined in Sec. 3.

The features of parameter τ may be inferred from Fig. 2a. The magnitude of τ is finite at $\gamma = 1$ and has its maximum value at $\gamma_{max} \simeq \frac{1}{2}\left[\ln N + 2 - \sqrt{\ln N(\ln N - 4)}\right]$, which implies $\gamma_{max} \approx 2.25$ for the networks with $N = 500$ nodes given in Figs. 1 and 2a, respectively. For Ising spin system determined by Hamiltonian (1) Eq. (15) establishes the second order PT from paramagnetic state ($S_z = 0$) to ferromagnetic one ($S_z \neq 0$), which occurs if normalized degree correlation function obeys the condition $\zeta_2 \geq \zeta_{2,c}$. For social network systems, such a PT indicates transformation from disorder to some ordered state. Hence, PT for the analyzed IM appears only due to finite size effects [17,18].

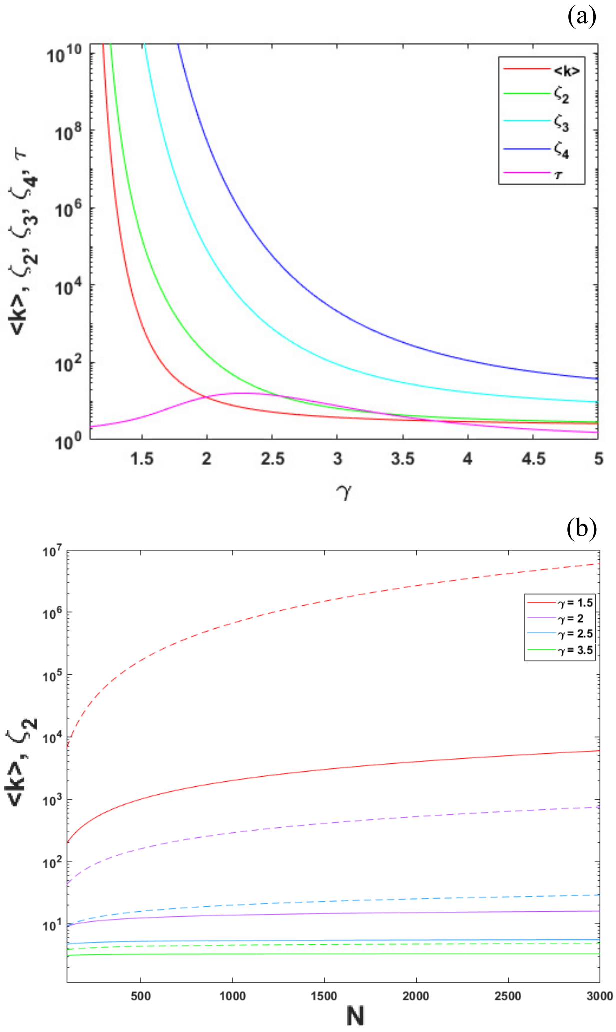

Fig. 2. Dependence of (a) k, τ, ζ_2, ζ_3 and ζ_4 on degree exponent γ for $k_{min} = 2$, $N = 500$, and (b) ζ_2 (dash line) and $\langle k \rangle$ (solid lines) on N for $k_{min} = 2$, and different values of γ. Both figures are plotted in logarithmic scale.

The main features of spin network are presented in Fig. 2b establishing behavior of average node degree k and correlation functions ζ_n ($n = 2,3,4$) and τ. These features are well distinguishable within anomalous ($1 < \gamma < 2$) and scale-free ($2 < \gamma < 3$) regimes, respectively. On the other hand, differences in ζ_n behavior vanish for large γ; in this limit the parameter τ responsible for some combination of degree correlation functions goes to unity (see Fig. 2b).

Remarkably, power-law degree distribution network parameter ζ_2 is the function of number of nodes N. The dependence of ζ_2 on N for various γ is demonstrated in Fig. 2b. As we can see, ζ_2 grows within the anomalous domain of γ. Therefore, all networks, which possess a growing number of hubs (decreasing γ), promote the occurrence of some ordered state.

In the presence of non-vanishing external (pump) field h for $\zeta_2 \to \zeta_{2,c}$ the order parameter may be obtained as:

$$S_z \simeq \left(\frac{3h}{4J\tau\zeta_{2,c}}\right)^{\frac{1}{3}} \simeq \left(\frac{3h}{4J\langle k\rangle_c}\right)^{\frac{1}{3}}, \tag{16}$$

where we assume $\tau \simeq 1$ and $\zeta_{2,c} \simeq \langle k\rangle_c$. The Eq. (16) represents the dependence of order parameter on degree exponent γ for scale-free networks; see Fig. 2.

5. CONCLUSION

In this work, we studied the features of the formation of a network structure with a power-law distribution of degrees of nodes using the adapted Barabasi-Albert algorithm under the condition of the Ising model. Since this work uses the mean field approximation, assuming that each spin is affected by the mean field from the other spins, we investigated how to build a more accurate model using this approximation.

We obtained an equation for the order parameter of the system determining the weighted average spin component. This expression allows us to establish the condition of the PT from the paramagnetic state to the ferromagnetic one. Critical values of PT temperatures are obtained in the presence and absence of an external field. The resulting equations establish a clear relationship between the temperature of the system and the statistical properties of the network. This research opens new possibilities for exploring macroscopic quantum states of matter for the development of new materials as well as for the development of new algorithms for information processing.

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Фазовые переходы в модели Изинга, определенной на сложных сетях

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Аннотация. В данной работе рассматривается модель Изинга, допускающая в системе спин-спиновое взаимодействие. Предполагается, что двухуровневые квантовые системы случайным образом расположены в N узлах сложной отожженной безмасштабной сети, описываемой моделью Барабаши-Альберта. Сеть характеризуется степенным распределением степеней узлов. В работе рассматривается подход среднего поля к системе Изинга. В рамках данного подхода система полностью характеризуется параметром порядка S_z и критической обратной температурой β, которая зависит от параметра ζ_2 как отношения первого и нормированного n-го моментов распределения степеней узлов. Было обнаружено, что при превышении параметром ζ_2 своего критического значения $\zeta_{2,c}$ происходит высокотемпературный фазовый переход, который может быть объяснен хабами и кластерами, которые появляются в безмасштабных сетях.

Ключевые слова: модель Изинга; фазовые переходы; сложные сеть; микрорезонаторы; наноматериалы